Sequence of transformations

Here's what we know so far. Relative to $\sin\theta$ and $\cos\theta$, the functions $\sin(\theta + c) + d$ and $\cos(\theta + c) + d$ are shifted

c units to the left if $c > 0$

c units to the right if $c < 0$

d units upwards if $d > 0$

d units downwards if $d < 0$

We also know that for functions $a\sin(b\theta)$ and $a\cos(b\theta)$

$|a|$ is the amplitude, and

if $|a| > 1$, the graph is stretched vertically by $|a|$.

if $0 < |a| < 1$, the graph is compressed vertically by $|a|$.

$2\pi/b$ is the period, and

if $0 < b < 1$, the graph is stretched horizontally by $1/b$.

if $b > 1$, the graph is compressed horizontally by b.

These horizontal and vertical shifts and horizontal and vertical stretches and compressions are all called "transformations."

In this lesson, we'll look at functions in the form

$$a\sin(b(\theta + c)) + d$$

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$$a\cos(b(\theta + c)) + d$$

where $a \neq 0$, $b > 0$, and c and d can be any real numbers.

When we graph functions that include these kinds of transformations, we'll use a systematic process and start with the graph of the basic sine or cosine function, and then gradually work our way through each transformation.

You always want to start from the "inside" and work your way toward the "outside." So for example, when you graph $a\sin(b(\theta + c)) + d$, you'll want to apply the transformation caused by c, then deal with b, then deal with a, and then apply d.

Example

What transformations are applied to take $\cos\theta$ to the given function?

$$\cos\left(\frac{3\theta}{2}\right) - 4$$

If we rewrite the argument of the cosine as

$$\frac{3\theta}{2} = \left(\frac{3}{2}\right)\theta = \left(\frac{3}{2}\right)(\theta - 0)$$

then we see that for the given function we have $a = 1$, $b = 3/2$, $c = 0$, and $d = -4$.

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Since $c = 0$ and $b = 3/2$, the first "real" transformation (i.e., the first transformation that brings about an actual change in the graph) is a horizontal compression. This decreases the period from 2π for $\cos\theta$ to

$$\frac{2\pi}{\frac{3}{2}} = 2\pi\left(\frac{2}{3}\right) = \frac{4\pi}{3}$$

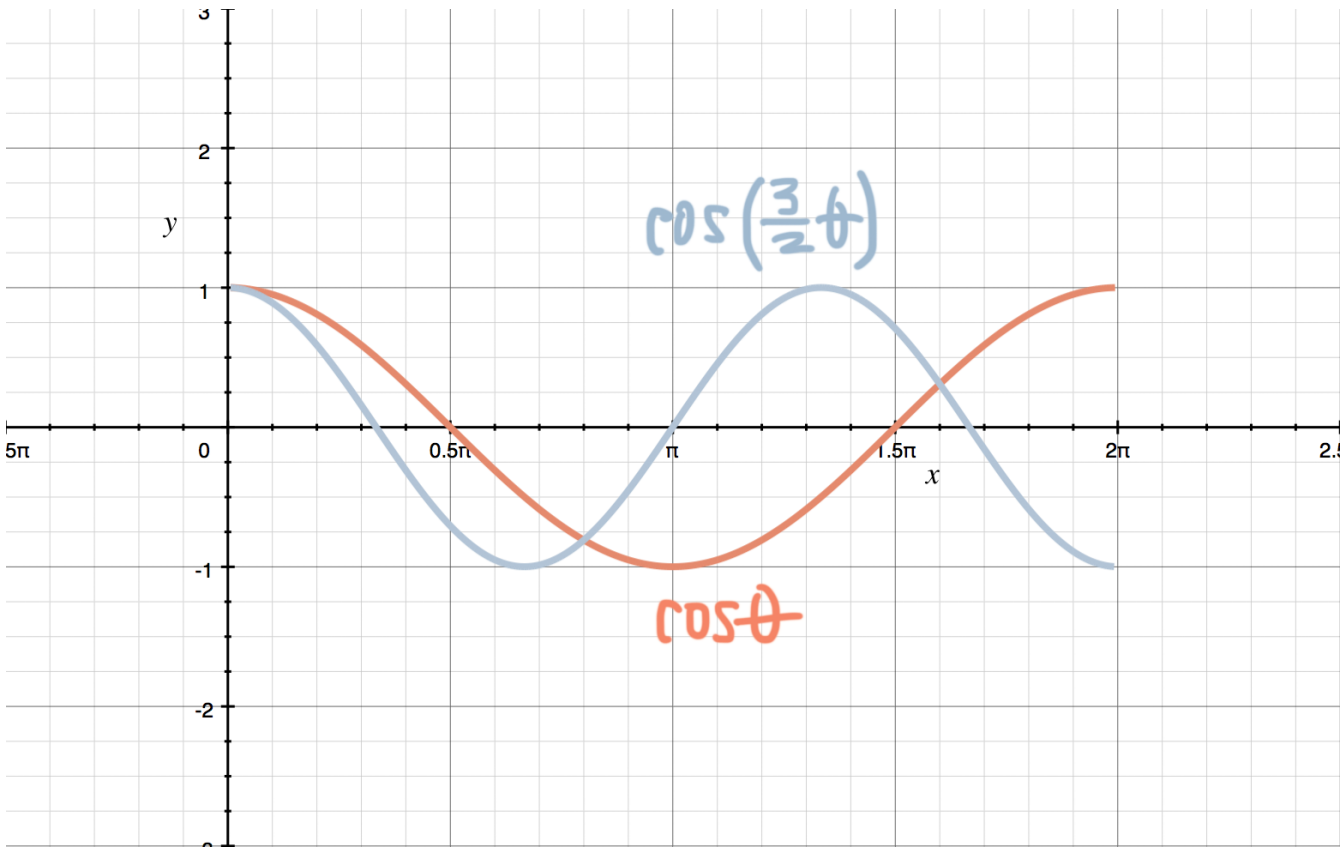
for

$$\cos\left(\frac{3}{2}\theta\right)$$

Which means that it increases the number of periods that fit into the interval $[0,2\pi)$ from 1 for $\cos\theta$ to $3/2 = 1.5$ for

$$\cos\left(\frac{3}{2}\theta\right)$$

We can sketch that transformation.



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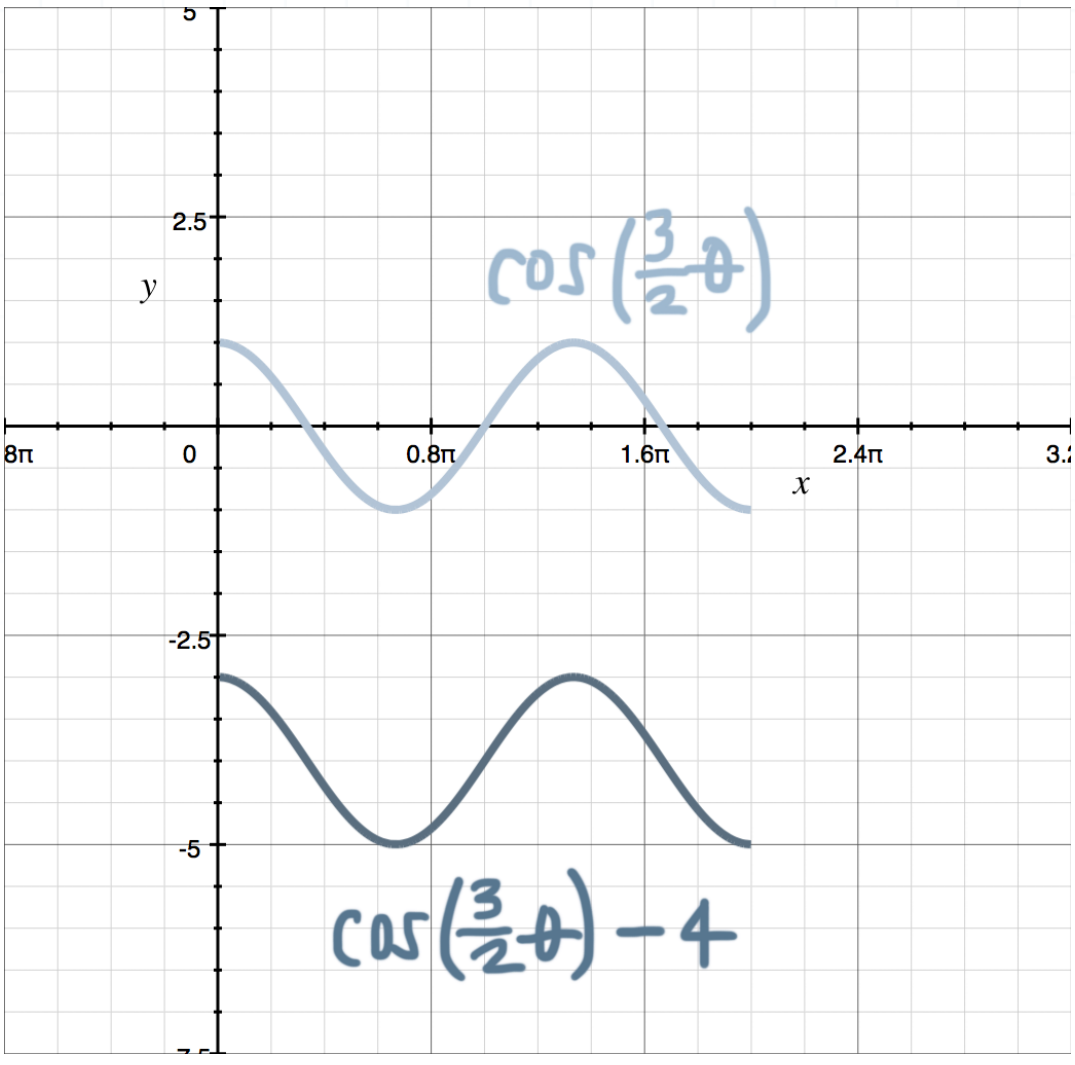
Since $a = 1$ and $d = -4$, the other "real" transformation is a vertical shift of -4. This takes the graph of

$$\cos\left(\frac{3}{2}\theta\right)$$

and shifts it 4 units downwards. So the graph of the given function after the shift looks like



Next, we'll look at a more complex function, one that exhibits three types of transformations.

Example

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What transformations are applied to take $\sin\theta$ to the given function?

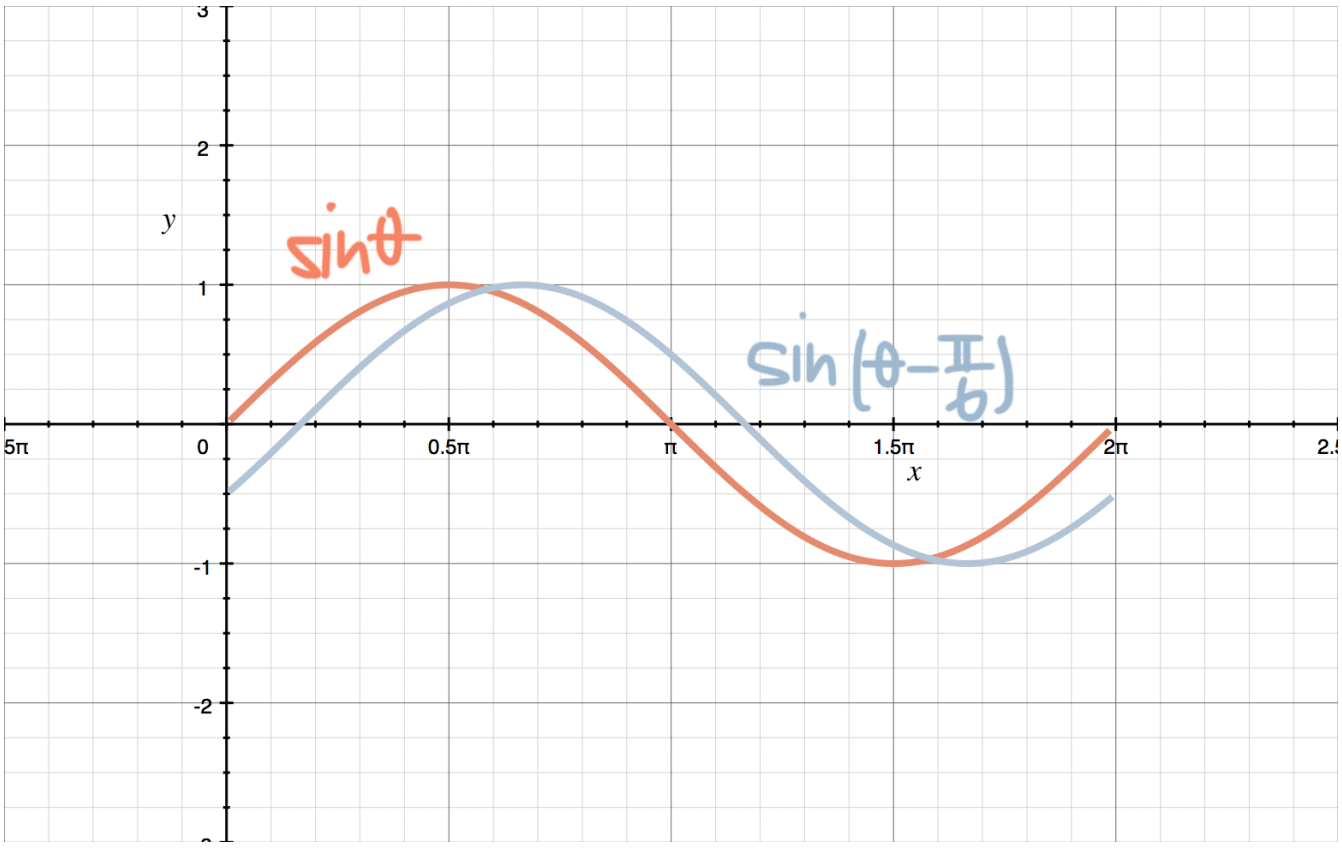
$$1.7\sin\left(2\left(\theta-\frac{\pi}{6}\right)\right)$$

For the function

$$1.7\sin\left(2\left(\theta-\frac{\pi}{6}\right)\right)$$

we have $a = 1.7$, $b = 2$, $c = -\pi/6$, and $d = 0$.

Since $c = -\pi/6$, the first "real" transformation is a horizontal shift. This gives us the graph of $\sin(\theta - (\pi/6))$, which is shifted $\pi/6$ units to the right.



Since $b = 2$, the second "real" transformation is a horizontal compression. This decreases the period from 2π for the function

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$$\sin\left(\theta-\frac{\pi}{6}\right)$$

to

$$\frac{2\pi}{2}=\pi$$

for the function

$$\sin\left(2\left(\theta-\frac{\pi}{6}\right)\right)$$

Which means it increases the number of periods that fit into the interval $[0,2\pi)$ from 1 for the function

$$\sin\left(\theta-\frac{\pi}{6}\right)$$

to 2 for the function

$$\sin\left(2\left(\theta-\frac{\pi}{6}\right)\right)$$

Graphing this transformation gives

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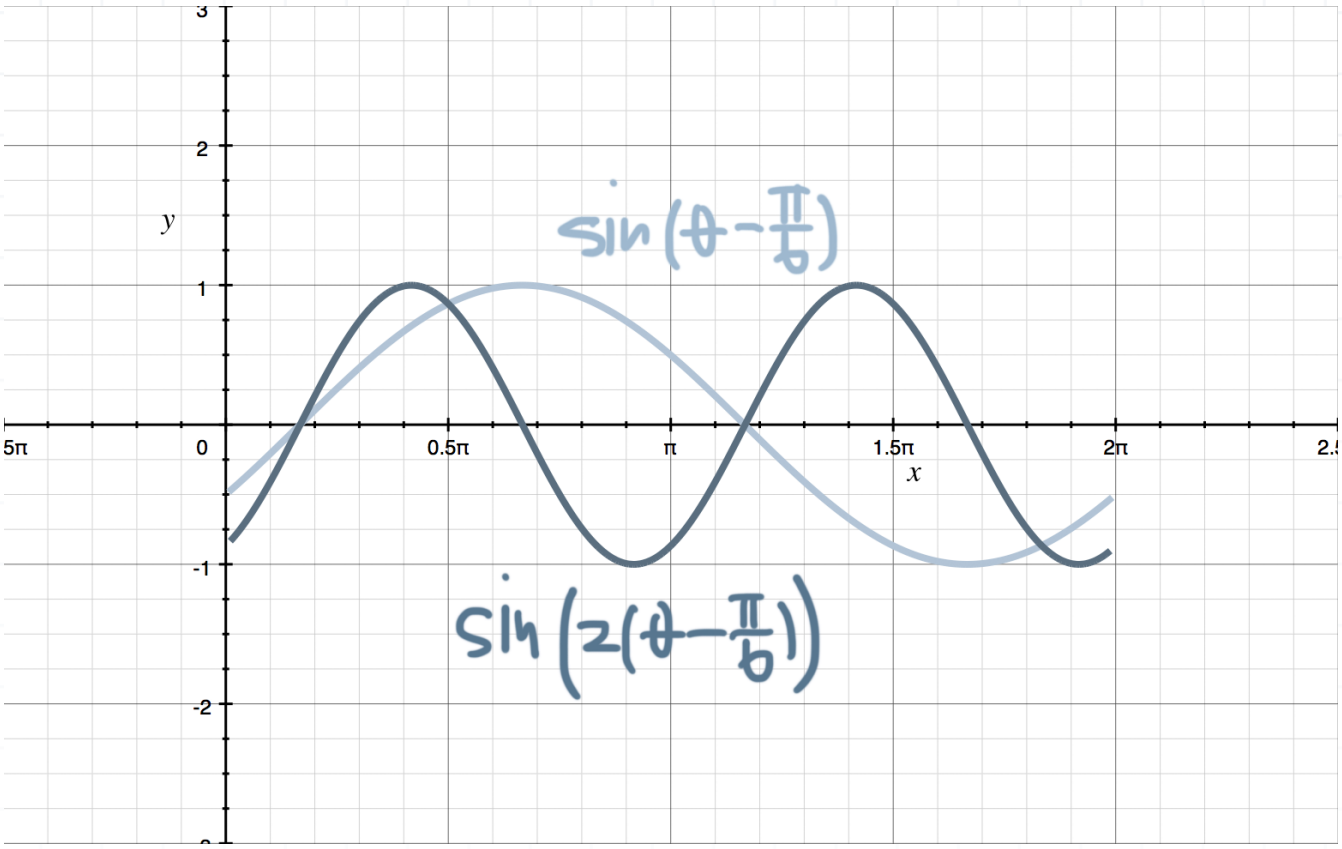

Since $a = 1.7$ and $d = 0$, the third and final "real" transformation is a vertical stretch of 1.7. This increases the amplitude from 1 for the function

$$\sin\left(2\left(\theta - \frac{\pi}{6}\right)\right)$$

to 1.7 for the function

$$1.7\sin\left(2\left(\theta - \frac{\pi}{6}\right)\right)$$

Graphing this transformation gives

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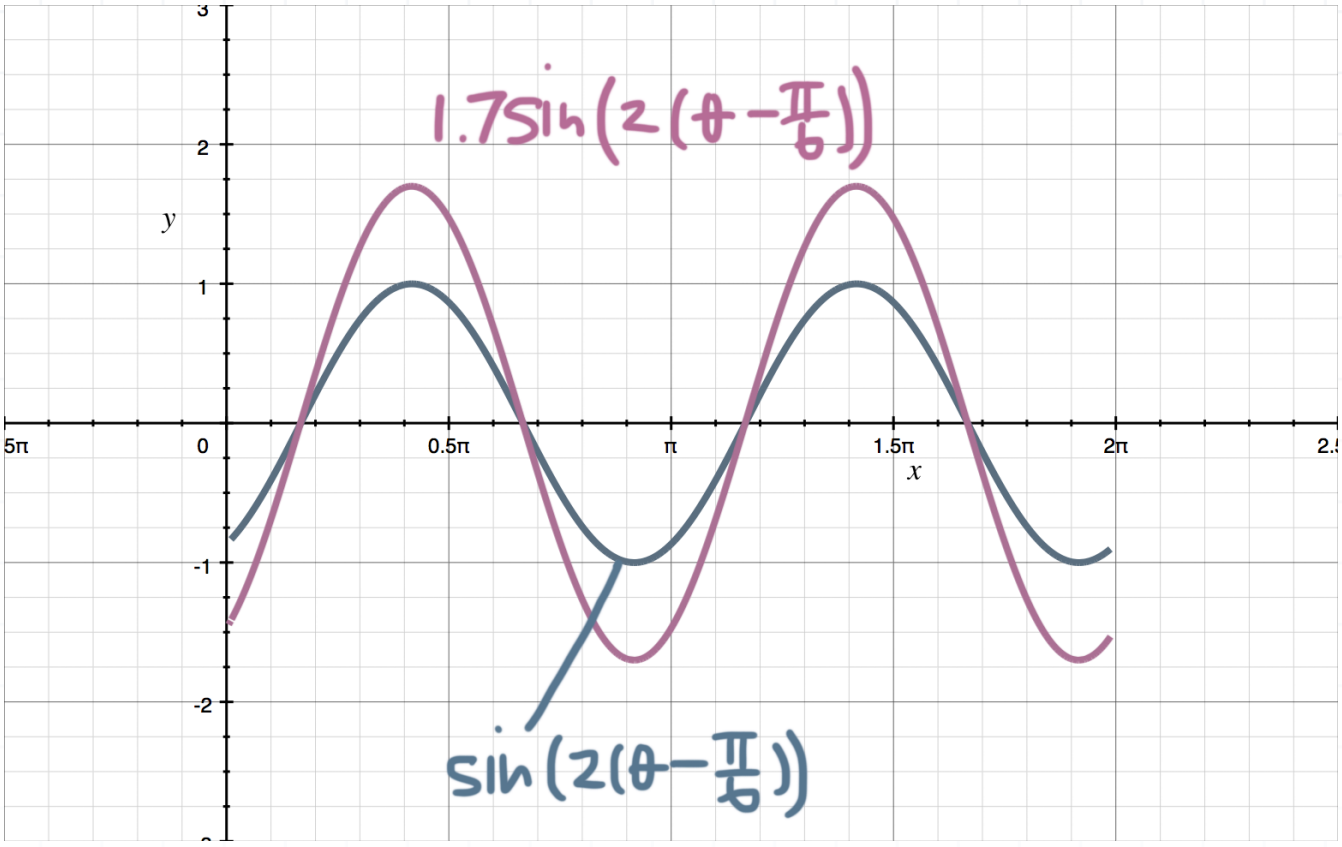

Finally, we'll look at a function whose graph undergoes four “real” transformations.

Example

What transformations are applied to take $\cos\theta$ to the given function?

$$-1.5\cos\left(\frac{3\theta}{4}+\frac{\pi}{4}\right)+2$$

We can immediately see that for the function

$$-1.5\cos\left(\frac{3\theta}{4}+\frac{\pi}{4}\right)+2$$

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we have $a = -1.5$ and $d = 2$. We'll rewrite the expression for this function in order to determine the values of b and c.

Factoring $3/4$ out of each term in

$$\frac{3\theta}{4} + \frac{\pi}{4}$$

we get

$$\frac{3\theta}{4} + \frac{\pi}{4} = \frac{3}{4}\left[\frac{4}{3}\left(\frac{3\theta}{4}\right) + \frac{4}{3}\left(\frac{\pi}{4}\right)\right] = \frac{3}{4}\left(\theta + \frac{\pi}{3}\right)$$

Therefore, the expression for the function

$$-1.5\cos\left(\frac{3\theta}{4} + \frac{\pi}{4}\right) + 2$$

can be rewritten as

$$-1.5\cos\left(\frac{3}{4}\left(\theta + \frac{\pi}{3}\right)\right) + 2$$

so $b = 3/4$ and $c = \pi/3$.

Since $c = \pi/3$, the first transformation is a horizontal shift. This gives us the graph of $\cos(\theta + (\pi/3))$, which is shifted $\pi/3$ units to the left.

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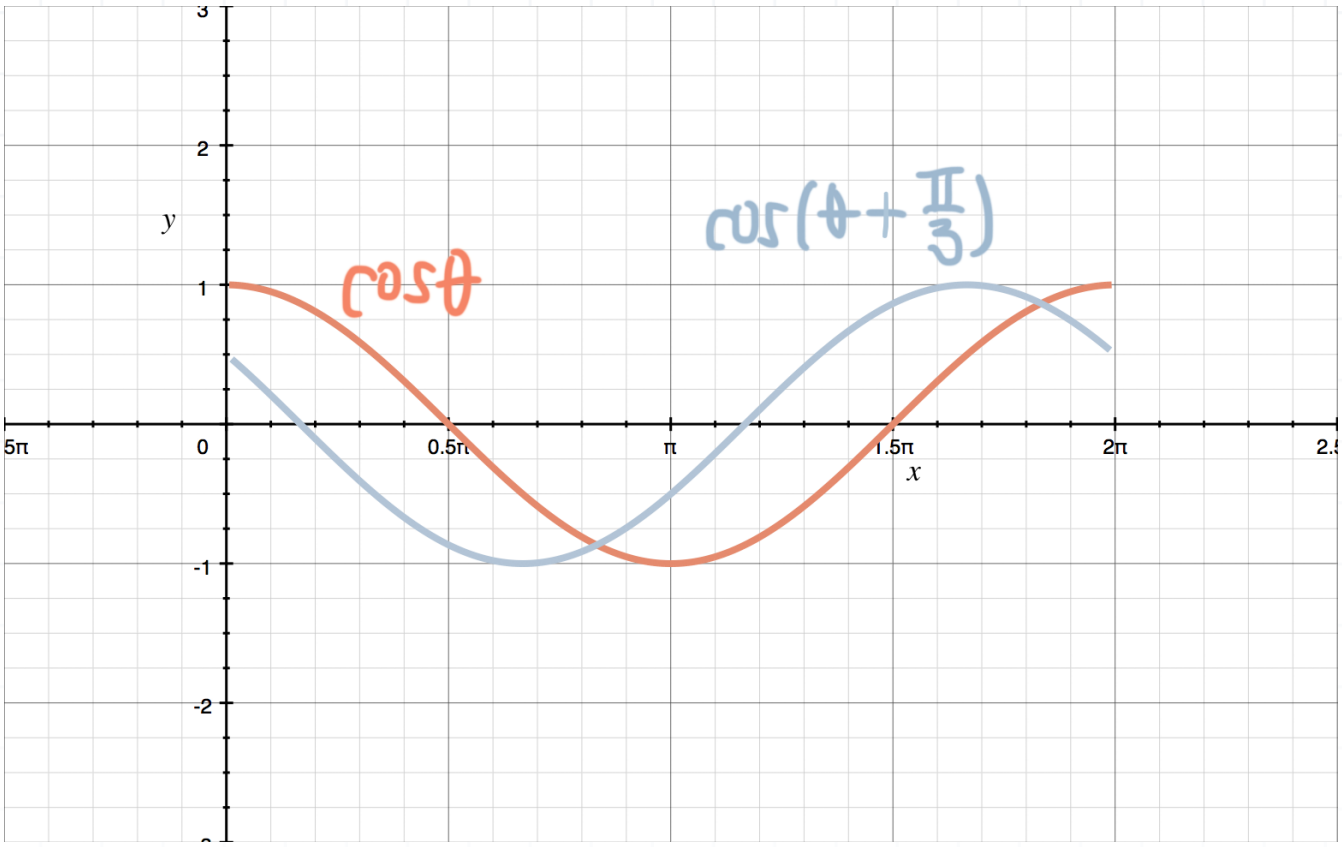

Since $b = 3/4$, the second transformation is a horizontal stretch. This increases the period from 2π for the function $\cos(\theta + (\pi/3))$ to

$$\frac{2\pi}{\frac{3}{4}} = 2\pi\left(\frac{4}{3}\right) = \frac{8\pi}{3}$$

for the function

$$\cos\left(\left(\frac{3}{4}\right)\left(\theta + \frac{\pi}{3}\right)\right)$$

Which means it decreases the number of periods that fit into the interval $[0,2\pi)$ from 1 for the function $\cos(\theta + (\pi/3))$ to $3/4$ for the function

$$\cos\left(\frac{3}{4}\left(\theta + \frac{\pi}{3}\right)\right)$$

Graphing that transformation gives

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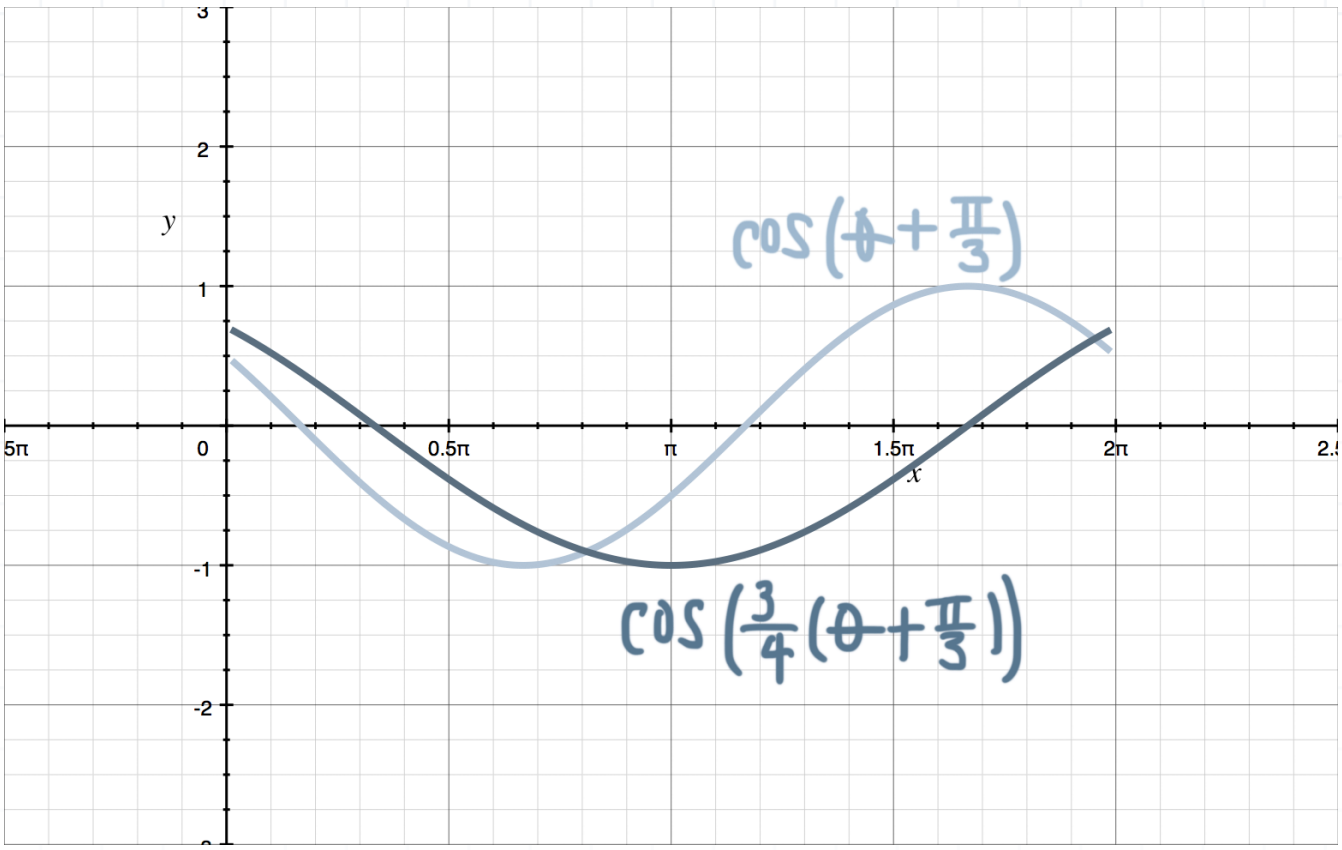

Since $a = -1.5$, the third transformation is a vertical stretch of 1.5 and a reflection in the horizontal axis. The vertical stretch increases the amplitude from 1 for the function

$$\cos\left(\frac{3}{4}\left(\theta+\frac{\pi}{3}\right)\right)$$

to 1.5 for the function

$$-1.5\cos\left(\frac{3}{4}\left(\theta+\frac{\pi}{3}\right)\right)$$

Graphing this transformation gives

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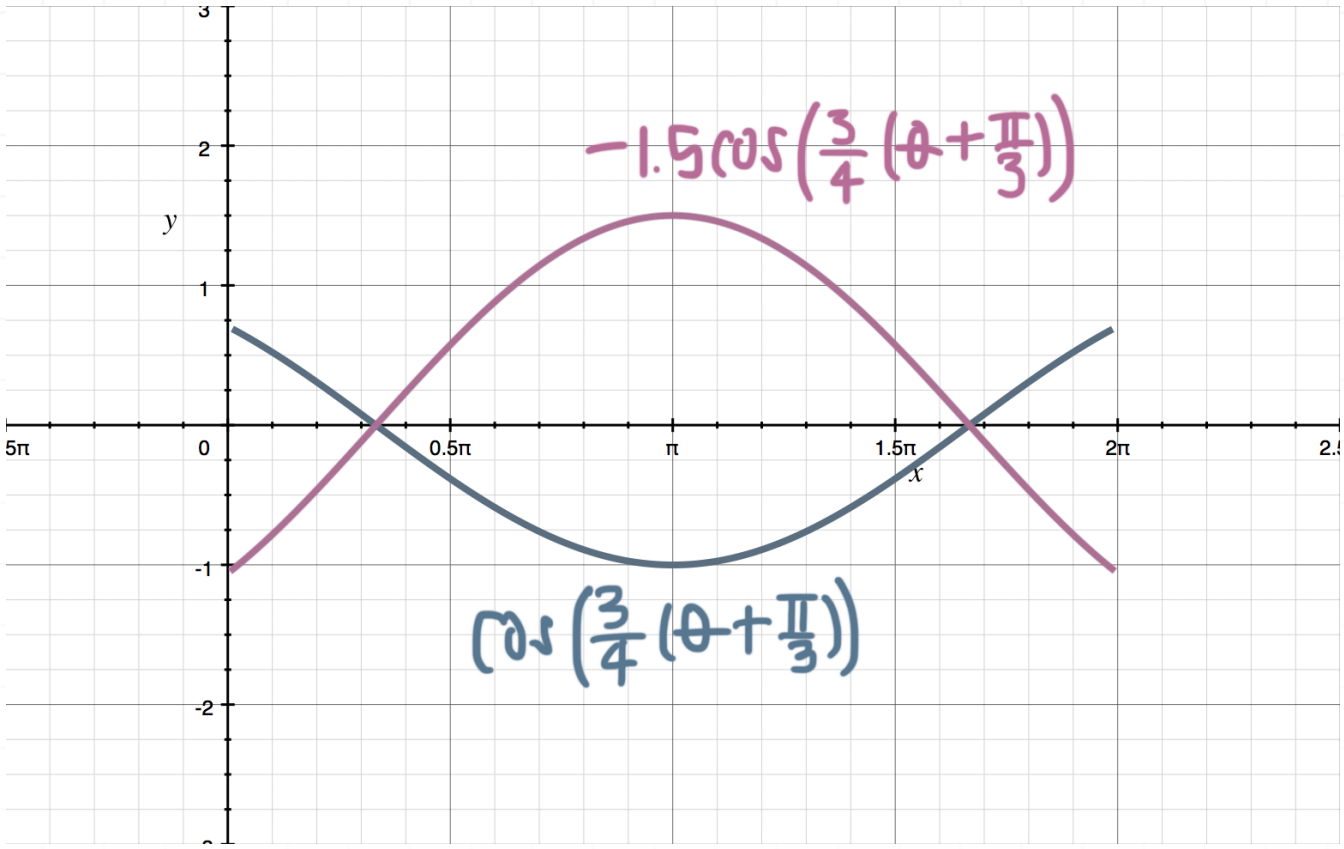

Finally, since $d = 2$, the fourth transformation is a vertical shift of 2. This takes the graph of

$$-1.5\cos\left(\left(\frac{3}{4}\right)\left(\theta+\frac{\pi}{3}\right)\right)$$

and shifts it 2 units upwards. Therefore, the final graph after this vertical shift is

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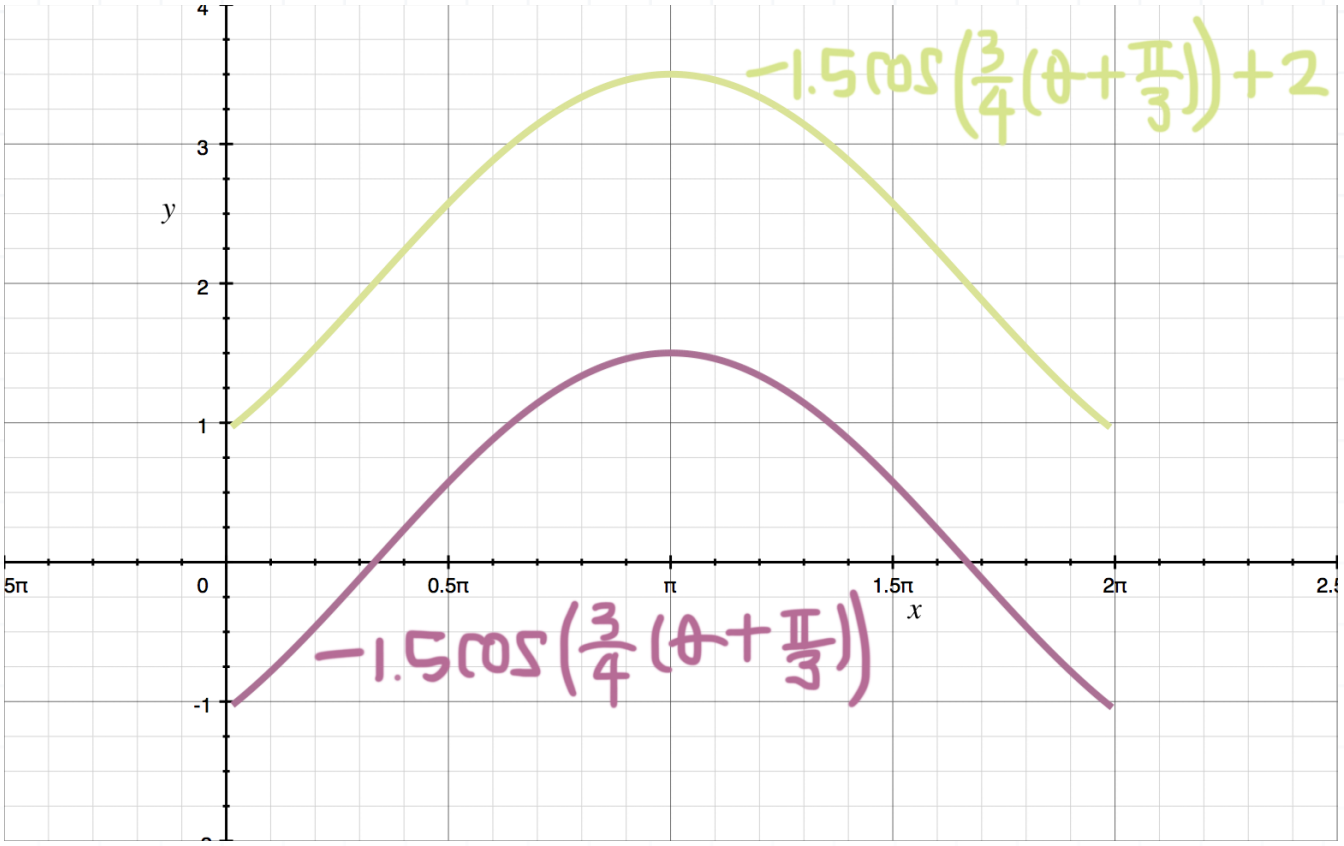

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